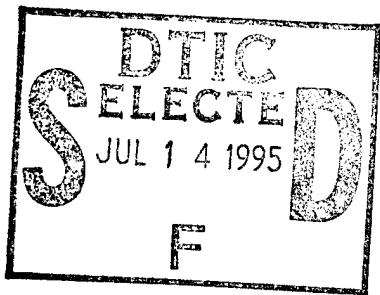


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TECHNICAL REPORT ARCCB-TR-95018

**IMPLEMENTING TENSOR ANALYSIS IN
Mathematica WITH ILLUSTRATIONS
FROM SCHWARZCHILD GRAVITATION**



L.V. MEISEL

MARCH 1995



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13. ABSTRACT (Maximum 200 words) Expressions that play roles in tensor analysis, such as Christoffel symbols and curvature tensors, are coded as <i>Mathematica</i> modules by straightforward transcription of their defining equations. The built-in functions can then be used to perform tensor analysis and numerical valuations. The utility of the <i>Mathematica</i> formulation is illustrated by examples from Schwarzschild gravitation.				
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INTRODUCTION .

Andrew and Fleming (ref 1) discussed the calculation of null geodesics for the Schwarzschild, Kerr-Newman, and Winicour metrics. Numerical computation employed FORTRAN code, which was produced by *Mathematica* (ref 2). The *Mathematica* code employed to generate the FORTRAN code for the neutral ($Q = 0$) Kerr-Newman geodesics was presented. Reference 1 provides a convincing demonstration of the utility of symbolic manipulation software for the development of error-free, complex FORTRAN (or C) code.

In this report, a general approach to problems in tensor analysis (ref 3) employing *Mathematica* is described. The standard expressions of tensor calculus are transcribed directly into *Mathematica* modules in Part I. The operation of the modules is illustrated as they are described by application to a simple two-dimensional curved space. The present formulation allows enumeration and simplification of complex tensor equations, employing built-in *Mathematica* functions (such as `Expand[]`, `Together[]`, or `Simplify[]`).

The code is applied to address some simple problems pertinent to Schwarzschild space-time in Part II. Employing the built-in *Mathematica* operator `Simplify[]`, the modules described produced FORTRAN coded Runge-Kutta geodesic equations for Kerr-Newman space-time in less than thirty minutes on a 386-based PC operating at 30 MHz. Of course, if simplification based on more efficient *Mathematica* operations (e.g., combinations of `Apart[]`, `Expand[]`, and `Together[]`) can be achieved, symbolic computation times can be dramatically reduced. Computation times are much shorter for the Schwarzschild geodesic equations.

PART I. TENSOR ANALYSIS IN *Mathematica*

Covariant Metric Tensor

The starting point for a typical problem in tensor analysis is the specification of the (covariant) metric tensor.

The operation of the *Mathematica* tensor analysis modules defined here is demonstrated by applications to geometry on the surface of a helicoid in R^3 having

$$g = \|g_{ij}\| = \begin{pmatrix} 1 & 0 \\ 0 & c^2 + x[1][s]^2 \end{pmatrix}$$

where the coordinates are $x[1][s]$ and $x[2][s]$ and c is constant. This g is referred to as the "example metric" in Part I. One encodes this g directly into *Mathematica* via

$$g = \text{DiagonalMatrix}[\{1, c^2 + x[1][s]^2\}];$$

Note that coordinates are given in the form $x[i][s]$. For the current analysis, one could simply use $x[i]$. However, as in Part II, one is often interested in derivatives with respect to the line element s , and we choose to include this dependence from the beginning.

Christoffel Symbols

The first step in a tensor problem is the computation of the Christoffel symbols. Christoffel symbols of the first kind are defined as

$$\Gamma_{i\alpha\beta} = \Gamma_{i\beta\alpha} = \frac{1}{2} \left(\frac{\partial g_{ai}}{\partial x^\beta} + \frac{\partial g_{i\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^i} \right)$$

Christoffel symbols of the second kind are given by

$$\Gamma_{\alpha\beta}^{\delta} = \sum_i g^{\delta i} \Gamma_{i\alpha\beta} = g^{\delta i} \Gamma_{i\alpha\beta}$$

where the last equality introduces "summation convention" which is adopted in this report and the contravariant metric tensor $g^{\delta i}$ is the inverse of the covariant metric, i.e.,

$$\delta_{\beta}^{\alpha} = g^{\alpha i} g_{i\beta}$$

The following *Mathematica* module defines the contravariant metric tensor and Christoffel symbols for arbitrary metric tensors in spaces of arbitrary dimension.

```

setup[g_]:=Module[{dim=Length[g]},
  ginv =Together[Inverse[g]];
  GAMMA =Array[g1,{dim,dim,dim}];
  gamma =Array[g2,{dim,dim,dim}];
  Do[ Do[ Do[GAMMA[[i,k,j]]=GAMMA[[i,j,k]]=
    Together[(1/2)(D[g[[i,j]],x[k][s]]+
      D[g[[k,i]],x[j][s]]-D[g[[j,k]],x[i][s]]),
      {i,dim},{k,j,dim},{j,dim}];
  Do[ Do[ Do[gamma[[i,k,j]]=gamma[[i,j,k]]=
    Together[Sum[ginv[[i,l]]GAMMA[[l,j,k]],{l,dim}]],
      {i,dim},{k,j,dim},{j,dim}]] ] ] ]

```

N. B., the partial derivative of a function $f[u,v,\dots,x,\dots]$ with respect to the variable x is computed via $D[f[u,v,\dots,x,\dots],x]$ in *Mathematica*. N.B., the *Mathematica* `Together[]` operator was effective in simplifying the resulting expressions for a number of "simple problems." Of course, the user could apply an other built-in (e.g., `Simplify[]`) or user-defined *Mathematica* operator.

After `setup[g]` is run, the contravariant metric tensor is returned as `ginv`, Christoffel symbols of the first kind are in the array `GAMMA`, i.e., $\Gamma_{ijk} = \text{GAMMA}[[i,j,k]]$, and Christoffel symbols of the second kind are in the array `gamma`, i.e., $\Gamma_{jk}^i = \text{gamma}[[i,j,k]]$.

To compute the Christoffel symbols and g^{-1} for the test metric, enter

setup[g].

Example: Display gamma for the example metric,

gamma

returns

$$\{\{0, 0\}, \{0, -x[1][s]\}, \{0, \frac{x[1][s]}{c + x[1][s]}, 0\}\},$$

$$\{\{0, 0\}, \{0, -x[1][s]\},$$

Exercise: Code a *Mathematica* module to compute covariant derivatives.

The Curvature Tensor

The curvature tensor can be computed directly from the $\Gamma_{\alpha\beta}^i$

$$B_{\alpha j k}^i = \frac{\partial \Gamma_{\alpha j}^i}{\partial x^k} - \frac{\partial \Gamma_{\alpha k}^i}{\partial x^j} + \Gamma_{\alpha j}^{\beta} \Gamma_{\beta k}^i - \Gamma_{\alpha k}^{\beta} \Gamma_{\beta j}^i$$

Transcription of the expressions for the $B_{\alpha j k}^i$ is straightforward,

```
curvature[gamma_,i_,a_,j_,k_]:=Module[{b,dim=Length[gamma]},
  Together[
    D[gamma[[i,a,j]],x[k][s]]-D[gamma[[i,a,k]],x[j][s]]+
    Sum[gamma[[b,a,j]]gamma[[i,b,k]]-
      gamma[[b,a,k]]gamma[[i,b,j]],{b,dim}]]
```

and full curvature tensor is returned by

```
curvature[gamma_]:=Module[{dim=Length[gamma]},
  B=Array[r1,{dim,dim,dim,dim}];
  Do[Do[Do[Do[B[[i,a,j,k]]=curvature[gamma,i,a,j,k],
    {i,dim}],{a,dim}],{j,dim}],{k,dim}];B]
```

The covariant form, $R_{hijk} = g_{ha} B_{i j k}^a$ is called the Riemann-Christoffel curvature tensor.

Example: Display the curvature tensor for the example metric

B = curvature[gamma]

returns

$$\begin{aligned} & \left\{ \left\{ \left\{ 0, 0 \right\}, \left\{ 0, 0 \right\} \right\}, \left\{ \left\{ 0, -\frac{c^2}{c^2 + x[1][s]^2} \right\}, \left\{ -\frac{c^2}{c^2 + x[1][s]^2}, 0 \right\} \right\} \right\}, \\ & \left\{ \left\{ 0, -\frac{c^2}{c^2 + x[1][s]^2} \right\}, \left\{ -\frac{c^2}{c^2 + x[1][s]^2}, 0 \right\} \right\}, \left\{ \left\{ 0, 0 \right\}, \left\{ 0, 0 \right\} \right\} \right\} \end{aligned}$$

The Ricci Tensor

The Ricci tensor plays an important role in relativity theory. It is defined as a contraction of the curvature tensor,

$$R_{ij} = B_{iaj}^a = \frac{\partial \Gamma_{ia}^\alpha}{\partial x^j} - \frac{\partial \Gamma_{ij}^\alpha}{\partial x^a} + \Gamma_{ia}^\beta \Gamma_{\beta j}^\alpha - \Gamma_{ij}^\beta \Gamma_{\beta a}^\alpha$$

Thus, to Print[] the components of $R_{ij} = B_{iaj}^a$, enter the *Mathematica* statement
 Do[Do[Print["R(",i,"",j,"") = ",Simplify[Sum[B[[a,i,a,j]],{a,2}]],
 {i,2}], {j,2}]

which yields

$$\begin{aligned} R(1,1) &= \frac{c^2}{c^2 + x[1][s]^2} \\ R(2,1) &= 0 \\ R(1,2) &= 0 \\ R(2,2) &= \frac{c^2}{c^2 + x[1][s]^2} \end{aligned}$$

N.B., Sum[*expression*,{i,n}] returns the sum of *expression* evaluated for i-values running from 1 to (the integer) n. Do[*expression*,{i,n}] works similarly.

We also transcribe separate *Mathematica* modules to compute the Ricci tensor,

```
Ricci[gamma_,i_,j_]:=Module[{m,n,dim=Length[gamma]},
Together[Sum[D[gamma[[m,i,m]],x[j][s]]-D[gamma[[m,i,j]],x[m][s]]+
Sum[gamma[[n,i,m]]gamma[[m,n,j]]-
gamma[[n,i,j]]gamma[[m,n,m]],{n,dim}},{m,dim}]]]
```

and

```
Ricci[gamma_]:=Module[{dim=Length[gamma]},
R=Array[r1,{dim,dim}];
Do[Do[R[[i,j]]=Ricci[gamma,i,j],{i,dim}},{j,dim}];R]
```

Example: Direct computation of Ricci tensor for the example metric.

```
Ricci[gamma]
```

returns

$$\left\{ \left\{ \frac{c^2}{(c^2 + x[1][s])^2}, 0 \right\}, \left\{ 0, \frac{c^2}{c^2 + x[1][s]^2} \right\} \right\}$$

which is identical to the results obtained by contraction of B, as expected.

Exercise: Compute the Christoffel symbols and Ricci tensor for

$$\|g_{ij}\| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

PART II. APPLICATION: THE SCHWARZCHILD METRIC

Development of a spherically-symmetric $\|g_{ij}\|$ consistent with the Einstein equations.
Assumption: $\|g_{ij}\|$ is spherically-symmetric. Assume that

$$g = \|g_{ij}\| = \begin{pmatrix} L[r] & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin^2(\theta) & 0 \\ 0 & 0 & 0 & M[r] \end{pmatrix}$$

$$= \begin{pmatrix} L[x[1][s]] & 0 & 0 & 0 \\ 0 & -x[1][s]^2 & 0 & 0 \\ 0 & 0 & -x[1][s]^2 \sin^2(x[2][s]) & 0 \\ 0 & 0 & 0 & M[x[1][s]] \end{pmatrix}$$

where the functions $L[r]$ and $M[r]$ are to be determined. Encode via,

`g=DiagonalMatrix[{L[x[1][s]],-x[1][s]^2,-(x[1][s]Sin[x[2][s]])^2,M[x[1][s]]}];`

Conventional Notation

Direct substitution is achieved in *Mathematica*, via the construction

`expression/.u->v`

which returns *expression* with occurrences of *u* replaced by *v*. Thus, to present output in {r,q,j,t}-form, define the following replacement rules:

`rt={x[1][s]->r,x[2][s]->q,x[3][s]->j,x[4][s]->t};`

N.B., *Mathematica* 2.1 does *not* have Greek fonts. We substitute *j* for ϕ and *q* for θ here and in the sequel with a text editor.

Compute the Christoffel symbols, etc.

`setup[g];`

The Einstein Equations

The Einstein equations require that $R_{ij} = 0$. Compute and display the Ricci tensor.

$$R = \text{Simplify}[\text{Ricci}[\text{gamma}]]/.rt$$

returns the Ricci tensor for the spherically-symmetric metric function,

$$\begin{aligned} & \left\{ \left\{ -\frac{L'[r]}{r L[r]} - \frac{L'[r] M'[r]}{4 L[r] M[r]} - \frac{M'[r]^2}{4 M[r]^2} + \frac{M''[r]}{2 M[r]}, 0, 0, 0 \right\}, \right. \\ & \left\{ 0, -1 - \frac{1}{L[r]} + \frac{r L'[r]}{2 L[r]^2} + \frac{r M'[r]}{2 L[r] M[r]}, 0, 0 \right\}, \\ & \left\{ 0, 0, (\sin[T] (-2 L[r] M[r] - 2 L[r]^2 M[r]^2 + \right. \\ & \left. r M[r] L'[r] - r L[r] M'[r])) / (2 L[r]^2 M[r]), 0 \right\}, \\ & \left. \left\{ 0, 0, 0, \frac{M'[r]}{r L[r]} - \frac{L'[r] M'[r]}{4 L[r] M[r]} - \frac{M'[r]^2}{2 L[r]^2} + \frac{M''[r]}{2 L[r]} \right\} \right\} \end{aligned}$$

Solution of the Einstein Equations

We seek functions $L[r]$ and $M[r]$ such that the Einstein equations are satisfied.

Step 1. Eliminate the $M''[r]$ terms from the 1,1 and 4,4 parts of $R_{ij} = R[[i,j]] = 0$.

$$\text{temp} = ((\text{Simplify}[(R[[1,1]]M[r[s]] - R[[4,4]]L[r[s]])]/.rt)) == 0)$$

returns

$$-\left(\frac{M[r] L'[r] + L[r] M'[r]}{r L[r]} \right) == 0$$

Step 2. Solve for $L[r]$ in terms of $M[r]$. Use

$$\|g_{ij}\| \xrightarrow{r \rightarrow \infty} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the *Mathematica* operator `DSolve[]` to obtain Rule0:

```
Rule0=ExpandAll[Flatten[DSolve[
  {L[Infinity]==-1,M[Infinity]==1,temp},L[r],r]]]
```

Mathematica issues a "warning:"

Solve::ifun:

Warning: Inverse functions are being used by Solve, so some solutions may not be found.

then returns the solution of interest

$$\{L[r] \rightarrow -(\frac{1}{M[r]})\}$$

Step 3. *Mathematica* will not apply the rule for $L[r]$ to replace $L'[r]$. Thus, one specifies a separate rule for $L'[r]$ and defines Rule1 as follows:

```
Rule1=Flatten[{Rule0,Solve[temp,L'[r]]/.Rule0}]/.r->x[1][s]
```

which returns

$$\{L[x[1][s]] \rightarrow -(\frac{1}{M[x[1][s]]}), L'[x[1][s]] \rightarrow \frac{M'[x[1][s]]}{M[x[1][s]]}\}$$

N.B., `Flatten[{{u},{v}}]` returns $\{u,v\}$, etc. Thus, the subrules comprising Rule1 will be applied consecutively.

Step 4. Use Rule1 and $R_{22} = 0$ to define the form of M . Express M in a rule (Rule2).

```
Rule2=Flatten[DSolve[
  0==Factor[Expand[R[[2,2]]/.Rule1]]/.rt,M,r]]]
```

which returns the rule governing the function M ,

$$\{M \rightarrow (1 + \frac{C[1]}{x[1][s]^2})\}$$

where C[1] is a constant of integration. The rule on the function works as expected, i.e.,

returns

$$M[x[1][s]]/.Rule2$$

$$1 + \frac{C[1]}{x[1][s]^2}$$

The Schwarzschild Metric

Applying Rule1 and Rule2, one obtains the form of the spherically-symmetric metric tensor consistent with $R = 0$:

returns

$$g_{\mu\nu} = g /. Rule1 /. Rule2 /. rt$$

$$\left\{ \left\{ -\frac{1}{1 + \frac{C[1]}{r^2}}, 0, 0, 0 \right\}, \left\{ 0, -r^2, 0, 0 \right\}, \left\{ 0, 0, -\left(r \sin[q] \right)^2, 0 \right\}, \left\{ 0, 0, 0, 1 + \frac{C[1]}{r^2} \right\} \right\}$$

As expected, this is the Schwarzschild metric form. The usual form of the metric tensor is obtained by identifying the integration constant C[1] with $-2 G M/c^2$, where G is the gravitational constant, M is the central mass, and c is the speed of light.

Exercise: Demonstrate that $R = 0$ for Schwarzschild gravitation.

Exercise: The geodesic trajectories are given by solutions of the geodesic equations

$$\frac{d^2 x[i][s]}{ds^2} + \Gamma_{\alpha\beta}^i \frac{dx[\alpha][s]}{ds} \frac{dx[\beta][s]}{ds} = 0$$

Derive the geodesic equations for

$$\|g_{ij}\| = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise: Derive the geodesic equations for the "example metric,"

$$\|g_{ij}\| = \begin{bmatrix} 1 & 0 \\ 0 & x[1][s]^2 + c^2 \end{bmatrix}$$

Exercise: Planetary orbits.

1. Derive the geodesic equations for the Schwarzschild metric.
2. Demonstrate that if $q[0] = p/2$ and $q'[0] = 0$, then $q[s] = p/2$.
 - a. Express the geodesic equations for $q[s] = p/2$.
3. Show that $r[s]^2 j'[s] = h = \text{constant}$.
4. Show that $(1 - (2G M/c^2) / r[s]) t'[s] = K = \text{constant}$.
5. Use the results of parts 1 through 4 and

$$1 = \frac{ds^2}{ds^2} = g_{ij} \frac{dx[i][s]}{ds} \frac{dx[j][s]}{ds}$$

$$= g_{ij} x[i]'[s] x[j]'[s] \rightarrow \sum_{i=1}^4 g_{ij} x[i]'[s]^2$$

to derive the *Mathematica* expression

$$1 == \frac{\frac{2}{K} - \frac{2}{h} (x[1])'[s]^2}{1 + \frac{C[1]}{x[1][s]} - \frac{2}{x[1][s]} + \frac{C[1]}{1 + \frac{C[1]}{x[1][s]}}}$$

6. Make the change of variables,

$$x[1][s] \rightarrow 1/u[x[3][s]],$$

and employ the condition derived in 3, to establish that

$$(x[1])'[s] \rightarrow -(h u'[x[3][s]])$$

and derive the equations:

$$1 = -\left(h^2 u[\phi]^2\right) + \frac{K^2}{1 + C[1] u[\phi]} - \frac{h^2 u'[\phi]^2}{1 + C[1] u[\phi]}$$

$$u'[\phi]^2 + u[\phi]^2 = -h^2 + \frac{-2 K C[1] u[\phi]}{h^2} - C[1]^3 u[\phi]^3$$

and

$$u''[\phi] + u''[\phi] = -\left(\frac{C[1]}{2 h}\right) - \frac{3 C[1] u[\phi]^2}{2}$$

The last equation for $u[\phi]$ ($= 1/r[j]$) frequently serves as the basis for a discussion of the advance of the perihelion of planetary orbits. See, for example, Crandall (ref 4). How should this approach be modified to treat photon trajectories?

Exercise: Compute the Christoffel symbols, curvature tensor, Ricci tensor, and geodesic equations for charge-free (i.e., $Q = 0$) Kerr-Newman gravitation. Demonstrate that $R = 0$ for the Kerr-Newman metric.

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2. Stephen Wolfram, *Mathematica, A System for Doing Mathematics by Computer*, Addison-Wesley, Redwood City, CA, 1991.
3. The expressions presented in this report may be found in any textbook that addresses Riemann geometry. See, for example, Harry Lass, *Vector and Tensor Analysis*, The Maple Press, York, PA, 1950.
4. Richard E. Crandall, *Mathematica for the Sciences*, Addison-Wesley, Redwood City, CA, 1991.

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